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Fractional discrete q-Fourier transforms

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Abstract
The discrete Fourier transform (DFT) matrix has a manifold of fractionalizations that depend on the choice of its eigenbases. One prominent basis is that of Mehta functions; here we examine a family of fractionalizations of the DFT stemming from q-extensions of this basis. Although closed expressions are given, many results of our analysis derive from numerical computation and display. Thus we suggest that the account of fractional Fourier transformations applied on signals as presented by other authors—typically of a centred rectangle function—may be biased because the support of the function lies in the central part of the domain only. The phase and amplitude of the whole fractional DFT matrices reveal the location of departures from the continuous kernel of the fractional Fourier integral transform, whose phase and constant amplitude are well known.

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1. Introduction

The fractionalization of the Fourier integral transform (FIT) was defined in 1937 by Condon [1] at the suggestion of von Neumann; since then, it has been rediscovered several times in the context of paraxial wave optics [2] and quantum oscillator systems [3]. The fractional powers of the FIT form a cyclic subgroup of the two-dimensional symplectic group represented by canonical integral transforms [4, Part IV], whose integral kernel was expressed as a bilinear generating function for Hermite–Gauss functions by Namias [5]. Fractional Fourier techniques have been used as tools for image processing in time and frequency, and other pairs of phase space coordinates [6]. To implement the fractional Fourier transform on finite data sets, however, several forms of regression to the $N \times N$ discrete Fourier transform (DFT) matrix have been marshalled to compute results so that the efficient FFT (fast Fourier transform) algorithm can be used [7–9].

Here we approach the fractionalization of the $N \times N$ DFT matrix by first recognizing that there is a large manifold of solutions to this task, which are determined by the choice of eigenbases of the DFT matrix. In [10] three bases were used to define fractional DFTs (FrDFT): the sampled Hermite–Gauss vectors, the Mehta basis [11] and Harper's basis [12]. The construction of generic FrDFTs is recounted in section 2. In this paper we extend the previous results in two directions: first, we introduce the q-extension of the Mehta eigenbasis of the DFT matrix [13] to define fractional discrete q-Fourier transform matrices (q-FrDFT); these form a one-parameter cyclic group of powers ν modulo 4, that coincide with the DFT matrices for integer ν. Second, we compare the q-FrDFTs with the FIT kernel itself, rather than its action on coherent states as in [10], or its rendering on rectangle signals only [14, 15]. Orthogonal bases lead to cyclic groups of unitary q-FrDFTs, while non-orthogonal ones—such as the original Mehta basis itself [16]—lead to groups of matrices that are unitary only for the integer ν.

In section 3, we review the Mehta basis, which is properly periodic, has an analytic form and possesses a natural q-extension that was recently proposed in [13]. In section 4, we compare the q-FrDFT matrices as approximations to the FIT kernel, whose salient property is to have an absolute value which is constant over the plane of its two coordinates, and a quadratic phase that is constant over hyperbolas. We evaluate whether this desideratum is furthered by the introduction of the q-parameter. Several conclusions in this regard are collected in the concluding section 5.

2. The discrete Fourier transform

The $N \times N$ DFT matrix is defined as

$$\mathbf{F} = \|F_{m,m'}\|, \qquad F_{m,m'} := \frac{1}{\sqrt{N}} \exp\left(-\mathrm{i}\frac{2\pi mm'}{N}\right), \tag{1}$$

with elements that are periodic in m, m' modulo N. This matrix is symmetric and unitary, so its inverse is its complex conjugate and $|\det \mathbf{F}| = 1$. Importantly, $\mathbf{F}^4 = \mathbf{1}$, so the DFT is a fourth root of unity; hence, its eigenvalues are (for $N > 4$) the four fourth roots of unity,

$$\varphi_0 := 1, \qquad \varphi_1 := -\mathrm{i}, \qquad \varphi_2 := -1, \qquad \varphi_3 := \mathrm{i}, \tag{2}$$

i.e. $\varphi_n = (-\mathrm{i})^n$ for $n = 0, 1, 2, 3$. The multiplicities N_{φ_n} of these eigenvalues have the following values for dimensions N depending on their residue modulo 4 [11]:

dimension	multiplicities N_{φ_n}					
$N=$	$\varphi_0 = 1$	$\varphi_1 = -\mathrm{i}$	$\varphi_2 = -1$	$\varphi_3 = \mathrm{i}$	tr $\mathbf{F}$	det $\mathbf{F}$
$4J$	$J+1$	J	J	$J-1$	$1+\mathrm{i}$	$-\mathrm{i}(-1)^J$
$4J+1$	$J+1$	J	J	J	1	$(-1)^J$
$4J+2$	$J+1$	J	$J+1$	J	0	$-(-1)^J$
$4J+3$	$J+1$	$J+1$	$J+1$	J	$-\mathrm{i}$	$\mathrm{i}(-1)^J,$

(3)

where $N = \sum_{n=0}^{3} N_{\varphi_n}$, tr $\mathbf{F} = \sum_{n=0}^{3} \varphi_n N_{\varphi_n}$ and det $\mathbf{F} = \prod_{n=0}^{3} (\varphi_n)^{N_{\varphi_n}}$.

There is a natural N-dimensional complex vector space $\mathcal{C}^N$ of 'discrete functions' (column vectors or 'signals') $\{f_m\}_{m=1}^N$ on which $\mathbf{F}$ and the FrDFT matrices act, describing the states of a discrete and finite system. This space is endowed with the usual inner product $(f, g) := \sum_{m=1}^{N} f_m^* g_m$ and norm $|f| := \sqrt{(f, f)}$. The DFT divides $\mathcal{C}^N$ into its four eigenspaces $\mathcal{C}^{N_{\varphi_n}}$, of dimensions N_{φ_n}, which are mutually orthogonal through the four projector matrices,

$$\mathbf{P}_{\varphi_n} := \frac{1}{4}\sum_{k=0}^{3}\varphi_n^{-k}\mathbf{F}^k, \qquad \mathbf{P}_{\varphi_n}\mathbf{P}_{\varphi_{n'}} = \delta_{n,n'}\mathbf{P}_{\varphi_n}. \tag{4}$$

From these, we can regain the DFT and its integer powers as

$$\mathbf{F}^{\nu} = \sum_{n=0}^{3}\varphi_n^{\nu}\,\mathbf{P}_{\varphi_n}, \qquad \varphi_n^{\nu} := \exp\left(-\mathrm{i}\frac{1}{2}\pi\nu n\right). \tag{5}$$

We shall now allow the powers ν in (5) to take arbitrary real values, noting that $\varphi_n^4 = \exp(-2\pi \mathrm{i}n) = 1$ implies that they are periodic modulo 4. These four projector matrices can be built with complete sets of eigenvectors of $\mathbf{F}$,

$$\mathbf{F}\mathbf{v}^{(\varphi_n,j)} = \varphi_n\mathbf{v}^{(\varphi_n,j)}, \qquad \mathbf{v}^{(\varphi_n,j)} = \left\{v_m^{(\varphi_n,j)}\right\}_{m=1}^{N} \in \mathcal{C}^{\varphi_n}, \tag{6}$$

where $\varphi_n \in \{1, -\mathrm{i}, -1, \mathrm{i}\}$ labels the orthogonal eigenspaces of $\mathbf{F}$ with $n \in \{0, 1, 2, 3\}$, and $j \in \{0, 1, \ldots, N_\varphi-1\}$ labels the N_φ vectors in each eigenspace.

Let us identify by V any basis that satisfies (6). It is not necessary that $\mathbf{v}^{(\varphi_n,j)} \perp \mathbf{v}^{(\varphi_n,j')}$ for $j \neq j'$, but only that there exists a dual basis of row N-vectors $\overline{\mathbf{v}}^{(\varphi,j)} = \left\{\overline{v}_m^{(\varphi,j)}\right\}_{m=1}^{N} \in \mathcal{C}^{N_\varphi}$, such that

$$\overline{\mathbf{v}}^{(\varphi,j)}\mathbf{v}^{(\varphi',j')} = \delta_{\varphi,\varphi'}\delta_{j,j'}, \qquad \sum_{j=0}^{N_\varphi-1}\mathbf{v}^{(\varphi,j)}\overline{\mathbf{v}}^{(\varphi,j)} = \mathbf{P}_\varphi. \tag{7}$$

To propose a discrete analogue we shall assign a single numeration for the N-vectors of the bases by using the compound index $k := 4j + n \in \{0, 1, \ldots, N-1\}$, which will serve to interleave the four φ_n eigenvalues in the same order as the energy eigenfunctions of the harmonic oscillator.

Following the construction with orthonormal bases in [10], we now build the FrDFT matrix $\mathbf{F}_{\mathrm{V}}^{\nu}$ using the V-basis vectors and their duals as

$$\left(\mathbf{F}_{\mathrm{V}}^{\nu}\right)_{m,m'} := \sum_{n=0}^{3}\sum_{j=0}^{N_{\varphi_n}-1} v_m^{(\varphi_n,j)} \exp\left[-\mathrm{i}\frac{1}{2}\pi(4j+n)\nu\right]\overline{v}_{m'}^{(\varphi_n,j)}. \tag{8}$$

Due to (7), these matrices have the desired properties:

$$\mathbf{F}_{\mathrm{V}}^{\nu_1}\mathbf{F}_{\mathrm{V}}^{\nu_2} = \mathbf{F}_{\mathrm{V}}^{\nu_1+\nu_2}, \qquad \mathbf{F}_{\mathrm{V}}^{0} = \mathbf{1} = \mathbf{F}_{\mathrm{V}}^{4}, \qquad \mathbf{F}_{\mathrm{V}}^{1} = \mathbf{F}. \tag{9}$$

They form a one-parameter cyclic Lie group of matrices

$$\mathbf{F}_{\mathrm{V}}^{\nu} = \exp\left(-\mathrm{i}\frac{1}{2}\pi\nu\mathbf{N}_{\mathrm{V}}\right), \tag{10}$$

with the generator

$$(\mathbf{N}_{\mathrm{V}})_{m,m'} = \sum_{n=0}^{3}\sum_{j=0}^{N_{\varphi_n}-1} v_m^{(\varphi_n,j)}(4j+n)\overline{v}_{m'}^{(\varphi_n,j)}, \tag{11}$$

which can be called the *number* matrix for $V \in \mathcal{C}^N$,

$$\mathbf{N}_{\mathrm{V}}\,\mathbf{v}^{(\varphi_n,j)} = (4j+n)\mathbf{v}^{(\varphi_n,j)}. \tag{12}$$

Let us call this subgroup of V-FrDFT matrices $\mathsf{C}_{\mathrm{V}} \subset \mathsf{SL}(N, \mathcal{C})$.

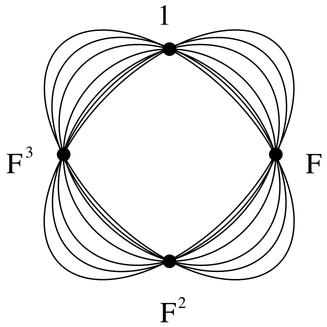

Figure 1. There is a manifold of bases V, each of which determines a circle of fractional Fourier matrices $\mathbf{F}_\mathrm{V}^\nu$ (depicted here as ellipses, since in two dimensions there is only one circle), passing through the unit **1**, the Fourier matrix **F**, and its integer powers. In the space of all complex $N \times N$ matrices of unit determinant, there are $2\sum_n N_{\varphi_n}^2 - 1$ real degrees of freedom in choosing the basis V.

The trace of the number matrix (11) is independent of V, and thus in any basis,

$$\mathrm{tr}\mathbf{N} = \sum_{n=0}^{3}\sum_{j=0}^{N_{\varphi_n}-1}(4j+n) = \begin{cases} 8J^2 - 2J + 1 & \text{for} \quad N = 4J, \\ 8J^2 + 2J & \text{for} \quad N = 4J+1, \\ 8J^2 + 6J + 2 & \text{for} \quad N = 4J+2, \\ 8J^2 + 10J + 3 & \text{for} \quad N = 4J+3, \end{cases} \tag{13}$$

and hence all FrDFT matrices (10) have the determinant

$$\det \mathbf{F}^\nu = \exp\left(-\tfrac{1}{2}\pi \mathrm{i}\,\nu\,\mathrm{tr}\,\mathbf{N}\right), \qquad \text{i.e. } |\det \mathbf{F}| = 1. \tag{14}$$

For $\nu = 1$, one recovers the values of $\det \mathbf{F}$ in the four cases of (3), so that (13) and (14) generalize the previously known results for any power ν of **F**.

We conclude that each complete basis V of the state space $\mathcal{C}^N$ thus determines its cyclic subgroup C_V of $N \times N\,V$-FrDFT matrices, parametrized by the power ν modulo 4. As submanifolds, the C_V's are circles in the $(2N^2-1)$-dimensional real space of $\mathsf{SL}(N, \mathcal{C})$. All these circles pass through the unit **1** and the DFT matrix **F**, and consequently through its integer powers $\mathbf{F}^2$ and $\mathbf{F}^3 = \mathbf{F}^{-1}$, but they are otherwise disjoint, as depicted in figure 1. The tangent to these circles at **1** is the number matrix $\mathbf{N}_\mathrm{V}$ in (11). The degrees of freedom we have to choose the circles is that of bases V, namely $2N_{\varphi_n}^2$ real parameters in each of the Fourier eigenspaces, minus the restriction that their determinant be 1. The only common eigenvectors of all C_V matrices are the vectors of the basis V.

When the chosen basis V is orthonormal (self-dual, so $\overline{v}_m^{(\varphi_n,j)} = v_m^{(\varphi_n,j)*}$), then the number matrix (11) is Hermitian and the V-FrDFT matrices $\mathbf{F}_\mathrm{V}^\nu$ in (8) are unitary, lying entirely in $\mathsf{SU}(N) \subset \mathsf{SL}(N, \mathcal{C})$; when V is not self-dual, then all but the integer powers ν will be non-unitary. When the vectors in an orthonormal basis V are also real, then the number matrix and all V-FrDFT matrices in C_V are symmetric. Other symmetries, such as those under $m \leftrightarrow N-m$ will be seen for the q-Mehta basis, to be introduced next.

3. The eigenbasis of *q*-Mehta functions

As was remarked in [10], the choice of the basis V for $\mathcal{C}^N$, and the numeration of its vectors $\mathbf{v}^{(\varphi_n,j)}$ by $k := 4j+n$, is in principle arbitrary. This brings to the fore the choice of 'good' bases, which should ensure that their V-FrDFTs have the same crucial properties as the continuous fractional FIT for the Fourier angle $\phi := \frac{1}{2}\pi\nu$ [4, 6],

$$\mathcal{F}^\nu f(x) := \int_{-\infty}^{\infty} \mathrm{d}x' F(x, x'; \nu) f(x'), \tag{15}$$

$$F(x, x'; \nu) = \frac{1}{\sqrt{2\pi \mathrm{i} \sin\phi}} \exp \mathrm{i}\left(-xx' \csc\phi + \frac{1}{2}(x^2+x'^2)\cot\phi\right) \tag{16}$$

$$= \sum_{k=0}^{\infty} \Psi_k(x)\, \mathrm{e}^{-\mathrm{i}\pi\nu/2}\Psi_k(x'). \tag{17}$$

The last expression is Namias' bilinear generating form [5] with the Hermite–Gauss (HG) eigenfunctions of the quantum harmonic oscillator,

$$\Psi_k(x) := \exp\left(-\tfrac{1}{2}x^2\right) H_k(x) \Big/ \sqrt{2^k k! \sqrt{\pi}}, \tag{18}$$

that are well known to be eigenfunctions of the FIT, $\mathcal{F}\Psi_k = (-\mathrm{i})^k \Psi_k$; their numeration by $k = 4j + n$ counts the energy quanta.

To propose discrete analogue of these formulae, one can consider HG functions sampled at N equidistant points with various scale factors [14, 15], but these values do not form eigenvectors of the DFT. Yet, Mehta [11] built eigenvectors of the $N \times N$ DFT through summing an infinite number of displaced copies of the HG functions as

$$\mu_k(x) := \sum_{\ell \in \mathcal{Z}} \Psi_k(s(x+\ell N)) = \mu_k(x+N), \qquad \begin{array}{l} s := \sqrt{2\pi/N}, \\ k \in \{0, 1, \ldots\}. \end{array} \tag{19}$$

These functions are periodic modulo N in the continuous parameter x, which served to give a very elegant proof of this fact, expanding $\mu_k(x)$ in a Fourier series of $\{\exp(\mathrm{i}s^2 nx)\}_{n\in\mathcal{Z}}$. The Fourier coefficients are found to be $\Psi_k(sn)$ times i^k, due to the self-reproduction of $\Psi_k(sx)$ under the FIT. Then, writing $n := \ell N + r$, the series is divided into a sum over $r \in \{1, 2, \ldots, N\}$ and a sum over $\ell \in \mathcal{Z}$, which returns $\mu_k(r)$ in (19). The resulting expression at the integer points $x = m$ modulo N closes the proof that the $\mu_k(m)$'s in (19) are eigenfunctions of the DFT with eigenvalues $(-\mathrm{i})^k$. Note that the Mehta functions are well defined and smooth for complex x, $\mathrm{Re}\, x^2 \geqslant 0$. The Mehta functions (and the q-Mehta functions, below) inherit the definite parity of the HGs, namely $\mu_k(-x) = (-1)^k \mu_k(x)$. It is sensible thus to choose the periodicity interval of the vector component index m of all vectors and matrices to be symmetric around $m = 0$, so that the properties under parity remain visible. Thus, for odd $N = 2K + 1$,

$$\mu_k(m),\ F^\nu_{m,m'}, \qquad m \in \{-K, -K+1, \ldots, 0, \ldots, K\} =: \mathcal{M}_N, \tag{20}$$

while for even $N = 2K$, the values of m would be taken over the half-integers between $\pm\left(K + \frac{1}{2}\right)$. For N odd, the Mehta N-vectors $\{\mu_k(m)\}_{k=0}^{N-1}$ are a linearly independent set; for N even, $k \in \{0, 1, \ldots, N-2, N\}$ is an independent set, as Mehta correctly conjectured [11]. In this paper we shall work with odd N's only, not to duplicate many of the formulae.

The Mehta basis has recently been subject to deformation by the parameter q [13], extending the Hermite polynomials in (18) to the q-Hermite $H_k(x|q)$ [17] and q^{-1}-Hermite $h_k(x|q)$ [18] polynomials. These can be defined by the three-level recurrence relations

$$H_{k+1}(x|q) = 2xH_k(x|q) - (1-q^k)H_{k-1}(x|q), \qquad H_0(x|q) = 1, \quad H_1(x|q) = 2x, \tag{21}$$

$$h_{k+1}(x|q) = 2xh_k(x|q) + (1-q^{-k})h_{k-1}(x|q), \qquad h_0(x|q) = 1, \quad h_1(x|q) = 2x, \tag{22}$$

for $k = 0, 1, 2, \ldots$ and $0 < q \leqslant 1$. These two polynomial families have the same lowest $k = 0$ and $k = 1$ members, and are related thereafter through $h_k(\sinh y|q) = \mathrm{i}^{-k} H_k(\sin \mathrm{i}y|q^{-1})$. When $q \to 1^-$, they become the standard Hermite polynomials through the (not obvious) limit relations

$$\lim_{q\to 1^-} \kappa^{-k} H_k(\sin \kappa x|q) = H_k(x) = \lim_{q\to 1^-} \kappa^{-k} h_k(\sinh \kappa x|q^{-1}), \tag{23}$$

$$\kappa := \sqrt{\left(-\tfrac{1}{2}\ln q\right)}, \qquad q = \exp(-2\kappa^2). \tag{24}$$

Their explicit polynomial form is

$$H_k(\sin y|q) := \mathrm{i}^{-k} \sum_{n=0}^{k} (-1)^n \begin{bmatrix} k \\ n \end{bmatrix}_q \mathrm{e}^{\mathrm{i}y(k-2n)}, \qquad 0 < q \leqslant 1, \tag{25}$$

$$h_k(\sinh y|q) := \sum_{n=0}^{k} (-1)^n \begin{bmatrix} k \\ n \end{bmatrix}_q q^{n(n-k)}\, \mathrm{e}^{y(k-2n)}, \qquad 0 < q \leqslant 1, \tag{26}$$

where the q-binomial and q-shifted factorials are [17]

$$\begin{bmatrix} k \\ n \end{bmatrix}_q := \frac{(q;q)_k}{(q;q)_n (q;q)_{k-n}}, \qquad (a;q)_k := \prod_{n=0}^{k-1} (1 - aq^n). \tag{27}$$

For numerical computation we have found the recurrence relations (21)–(22) to be much preferable to the direct computation by (25) and (26). With these q-Hermite polynomials we now define two sets of functions, evaluated on the same discrete points $x = sm$, $m \in \mathcal{M}_N$ and the scale $s^2 = 2\pi/N$ as in (19), with a common Gaussian factor. These are [13]

$$\Psi_k(m|q) := \exp\left(-\tfrac{1}{2}(sm)^2\right) H_k(\sin(\kappa sm)|q) \Big/ \sqrt{2^k k! \sqrt{\pi}}, \tag{28}$$

$$\psi_k(m|q) := \mathrm{i}^k \exp\left(-\tfrac{1}{2}(sm)^2\right) h_k(\sinh(\kappa sm)|q) \Big/ \sqrt{2^k k! \sqrt{\pi}}, \tag{29}$$

for $0 < q \leqslant 1$. And then, as Mehta did, we build the sums of integer-displaced q-HGs:

$$M_k(m|q) := \sum_{\ell\in\mathcal{Z}} \Psi_k(m+\ell N|q), \qquad \mu_k(m|q) := \sum_{\ell\in\mathcal{Z}} \psi_k(m+\ell N|q). \tag{30}$$

We call this q-deformation of the HG harmonic oscillator wavefunctions (28) and (29) q-HG functions, $\{\Psi_k(m|q)\}_{k=0}^{N-1}$ and $\{\psi_k(m|q)\}_{k=0}^{N-1}$. They are *two* sets of functions that are *not* separately eigenvectors of the Fourier transform, but instead exhibit the following very important reciprocal relation with the DFT matrix (1) found by Atakishiyev [19, 20]:

$$\sum_{m'\in\mathcal{M}_N} F_{m,m'} M_k(m'|q) = q^{k^2/4} \mu_k(m|q), \tag{31}$$

$$\sum_{m'\in\mathcal{M}_N} F_{m,m'} \mu_k(m'|q) = (-1)^k q^{-k^2/4} M_k(m|q). \tag{32}$$

As in the case of the original Mehta functions, these relations stem from corresponding Fourier integral transform relations satisfied by (28) and (29). Fourier eigenvectors can be now built as

$$\Phi_k(m|q) := \tfrac{1}{2}\left(q^{-k^2/8} M_k(m|q) + \mathrm{i}^{-k} q^{+k^2/8} \mu_k(m|q)\right). \tag{33}$$

These we call q-Mehta functions of number $0 \leqslant k \leqslant N-1$, defined on the points $m \in \mathcal{M}_N$, and extendable to regions of the complex plane where the sums converge. They are real functions, periodic in m modulo N, with definite parity $\Phi_k(-m|q) = (-1)^k \Phi_k(m|q)$, and are eigenvectors of the DFT matrix,

$$\sum_{m' \in \mathcal{M}_N} F_{m,m'} \Phi_k(m'|q) = (-\mathrm{i})^k \Phi_k(m|q). \tag{34}$$

We note that these functions are neither orthogonal nor normalized over $\mathcal{C}^N$ (yet).

The q-Mehta functions (33) provide the vector bases to build the fractional q-Fourier transforms (q-FrDFTs). The q-Mehta bases are conjectured to be complete; this is surely true for N odd, where numerical computation confirms intuition; it may be true or not when N is even.

To understand the structure of such non-orthogonal bases, we have found it very convenient to use density plots of the ordered vector elements $\Phi_k(m|q)$ for $0 \leqslant k \leqslant N-1$ and $m \in \mathcal{M}_N$, as was done in [10]. Thus in figure 2 we show the q-Mehta bases for three values of the deformation parameter q, starting with the $q = 1$ undeformed basis of original Mehta functions. The parabolic pattern of maxima widens as do the classical oscillator turning points, $\sim\sqrt{k}$, and there is a shortening of the intervals between sign alternations by $\sim 1/\sqrt{k}$. The two states $k = 0$ and 1 are q-independent as we saw above, and their lowest neighbours are relatively insensitive to the value of q. For $q < 1$ and higher k's, the patterns reveal two superposed parabola-like structures which are due to the sum of two terms in (33), the inner one stemming from $M_k(m|q)$ and the outer one from $\mathrm{i}^\kappa \mu_k(m|q)$. In the same figure we also use density plots to represent the non-orthogonality of a given real basis through building the matrix of their overlaps, $(\Phi_k(m|q), \Phi_{k'}(m|q))$. Were the basis orthonormal, a Kronecker $\delta_{k,k'}$ would be on the diagonal; we see this for the smaller k and k''s, but for larger k's (belonging to the same Fourier eigenspace) off-diagonal values appear; these increase as we leave the $q = 1$ undeformed case. For $q = 1$, the overlaps have been given by Ruzzi [16] in analytic form with Jacobi ϑ-functions.

We may ask if the q-Mehta functions are in some sense wavefunctions of a discrete oscillator. For example, the Harper basis is defined as the eigenbasis of the Harper oscillator [10, 12], defined by its Hamiltonian $\frac{1}{2}(\boldsymbol{\Delta} + \widetilde{\boldsymbol{\Delta}})$, where $\boldsymbol{\Delta}$ is the second-difference matrix corresponding to the kinetic energy $-\frac{1}{2}\boldsymbol{\Delta}$, and $-\frac{1}{2}\widetilde{\boldsymbol{\Delta}}$ is its diagonal Fourier transform, representing the discrete and periodic Harper potential $V_H(m) := 2(1 - \cos s^2 m), m \in \mathcal{M}_N, s^2 = 2\pi/N$. In continuous quantum mechanics the ground state of a system determines its potential through

$$\left(-\tfrac{1}{2}\nabla^2 + V(x)\right)\phi_0(x) = E_0\phi_0(x) \quad \Rightarrow \quad (V(x) - E_0) = \nabla^2\phi_0(x)/2\phi_0(x). \tag{35}$$

In our discrete system, the ground state $k = 0$ is given by a Jacobi theta function [13],

$$\pi^{1/4}\Phi_0(m|q) = \sum_{\ell \in \mathcal{Z}} \exp\left(-\frac{\pi}{N}(m+\ell N)^2\right) = \mathrm{e}^{-\pi m^2/N}\vartheta_3(\pi \mathrm{i} m, \mathrm{e}^{-\pi N}), \tag{36}$$

which has no zeros in $m \in \mathcal{M}_N$. Thus it can serve to similarly define an *equivalent* potential [21, 22] through

$$V_\Phi(m) := (\Phi_0(m+1) + \Phi_0(m-1))/\Phi_0(m) + \text{const}. \tag{37}$$

This potential is periodic in the position label m, and can be analytically continued or computed over $m \in \Re$; it is shown in figure 3 where indeed it resembles that of a harmonic oscillator. The potential (37) is independent of the deformation parameter q, suggesting that this parameter is not of dynamical import, but only embodies a choice of basis.

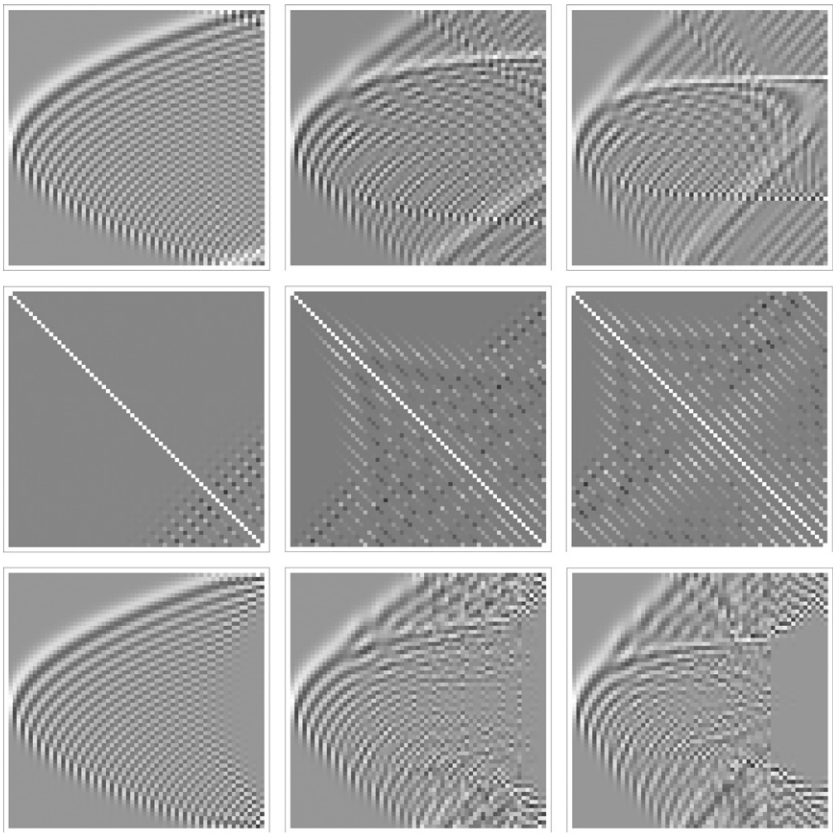

Figure 2. The q-Mehta bases for $N = 63$. Top row: density plots of the basis vectors $\Phi_k(m|q)$ (in each, columns are numbered by $k = 0, 1, \ldots, N-1 = 62$ and rows by $m \in \mathcal{M}_{63} = \{-31, \ldots, 31\}$). From left to right, for $q = 1$ (undeformed), 0.9 and 0.8. Middle row: overlaps $(\Phi_k(q), \Phi_{k'}(q))$ (rows and columns numbered by k, k') for the same values of q. Bottom row: the orthonormalized q-Mehta bases (compared with the top row).

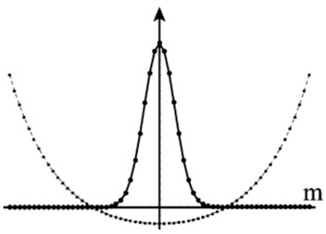

Figure 3. The q-Mehta equivalent potential (37) for $N = 63$ points (values at integer m's joined by a thick line, scaled to fit the figure), and the Mehta ground state $\Phi_0(m)$ in (36) (values joined by a dotted line). Both are independent of the deformation parameter q.

4. The integral vis-á-vis discrete transforms

As we stated in the introduction, our aim is to compare the fractional DFTs obtained from the q-Mehta functions, with the fractional FITs $\mathcal{F}^\nu$ in (15)–(17). The evident properties of the integral kernel $F(x, x'; \nu)$, $\nu = \phi/\frac{1}{2}\pi$, are that, for any $\nu \neq 0$, 2 modulo 4, the absolute value

$|F(x, x'; \nu)| = 1/\sqrt{(2\pi|\sin\phi|)}$ is constant over the x–x' plane, while its phases are constant on the hyperbolas

$$(x^2 + x'^2)\cot\phi - 2xx'\csc\phi = \text{constant}. \tag{38}$$

These are centred hyperbolas whose asymptotes lie in the first and third quadrants, given by the lines $(x/x')^{\pm 1} = \sec\phi \pm \tan\phi > 0$. For example, the square root of the FIT, $\nu = \frac{1}{2}$, has asymptotes $(x'/x)^{\pm 1} = \sqrt{2} \pm 1 =: \tan\alpha_{\pm}$, which lie at $\alpha_- = 22°30'$ and $\alpha_+ = 90° - \alpha_- = 67°30'$. For the DFT matrix $\nu = 1$ the asymptotes become the $\pm x$ and $\pm x'$ axes, while for $\nu \to 0$, both asymptotes coalesce at 45° as the kernel becomes $\delta(x - x')$.

Since the fractional FITs are unitary for all ν, we should use orthonormal bases to construct correspondingly unitary FrDFT matrices. Since neither the Mehta nor q-Mehta bases are orthogonal, we should subject the vectors of the bases $\Phi_k(m|q)$ in (33) to a Schmidt orthogonalization procedure within each of the four Fourier eigenspaces, in the natural order given by $k = 4j + n, n \in \{0, 1, 2, 3\}, 0 \leqslant j \leqslant N_{\varphi_n} - 1$. This was done by Pei *et al* [14, 15] for the sampled HG-functions. Henceforth, the computation must be basically numerical because the Schmidt process and final normalization produce rather unmanageable formulae. Yet, the process is unique, and thus out of the q-Mehta basis (33) we build the orthonormalized q-Mehta bases $\Phi_k^\circ(m|q)$, whose elements are real, to define the q-FrDFT matrix $\mathbf{F}_q^\nu$ (8), with elements

$$\left(\mathbf{F}_q^\nu\right)_{m,m'} := \sum_{k=0}^{N-1} \Phi_k^\circ(m|q) \exp\left(-\mathrm{i}\frac{1}{2}\pi k\nu\right) \Phi_k^\circ(m'|q). \tag{39}$$

In figures 4, we show density plots of the matrix elements of the q-FrDFT matrices (39), built with orthonormalized q-Mehta bases, for three values of q, starting with the undeformed case $q = 1$. We have checked of course that for any value of q one recovers the DFT matrix at $\nu = 1$, and that the product of the q-FrDFT matrices $\mathbf{F}_q^{\nu_i}$ yields another such matrix with the sum of powers of the factors. These matrices are unitary and symmetric; hence their inverse is their complex conjugate as for the fractional FIT kernel, and the matrix elements are periodic in m, m' modulo N. We see in the figure that the absolute values of the matrix elements $\left(\mathbf{F}_q^\nu\right)_{m,m'}$ are approximately constant only over a central region of roughly elliptical shape in the m–m' plane, which diminishes in size with the parameter q. The phases follow the pattern of hyperbolas (38), that appear to close as circles, which are particularly visible for $\nu = 1$.

The q-FrDFT of a signal vector $\mathbf{f}$—a rectangle is commonly used in the literature as a test function [6, 14, 15]—can be seen as the matrix product of the square density plots of $\mathbf{F}_q^\nu$ in figures 4 on an $N \times 1$ column vector $\{f_m\}$ represented by a corresponding density plot. As long as the significantly non-zero portion of $\mathbf{f}$ is multiplied by the plateau region of the matrix, the result will approximate the fractional FIT of the corresponding 'continuous' rectangle function. In figures 5 we show the application of various q-FrDFTs to a rectangle signal.

5. Concluding remarks

We have introduced the fractional discrete q-Fourier transform matrices, $\left(\mathbf{F}_q^\nu\right)_{m,m'}$ in (39), using q-extensions of the Mehta basis of DFT eigenstates. These matrices form cyclic subgroups within $\mathsf{SL}(N, \mathcal{C})$ that pass through the unit matrix $\mathbf{1}$ and the DFT matrix $\mathbf{F}$ (and its integer powers); the matrices will be unitary only when the bases are orthonormal, so we subjected them to the Schmidt process. As an informative aid, we presented density plots of the basis vectors (figure 2), and of the absolute value and phase of the q-Fourier transform matrix

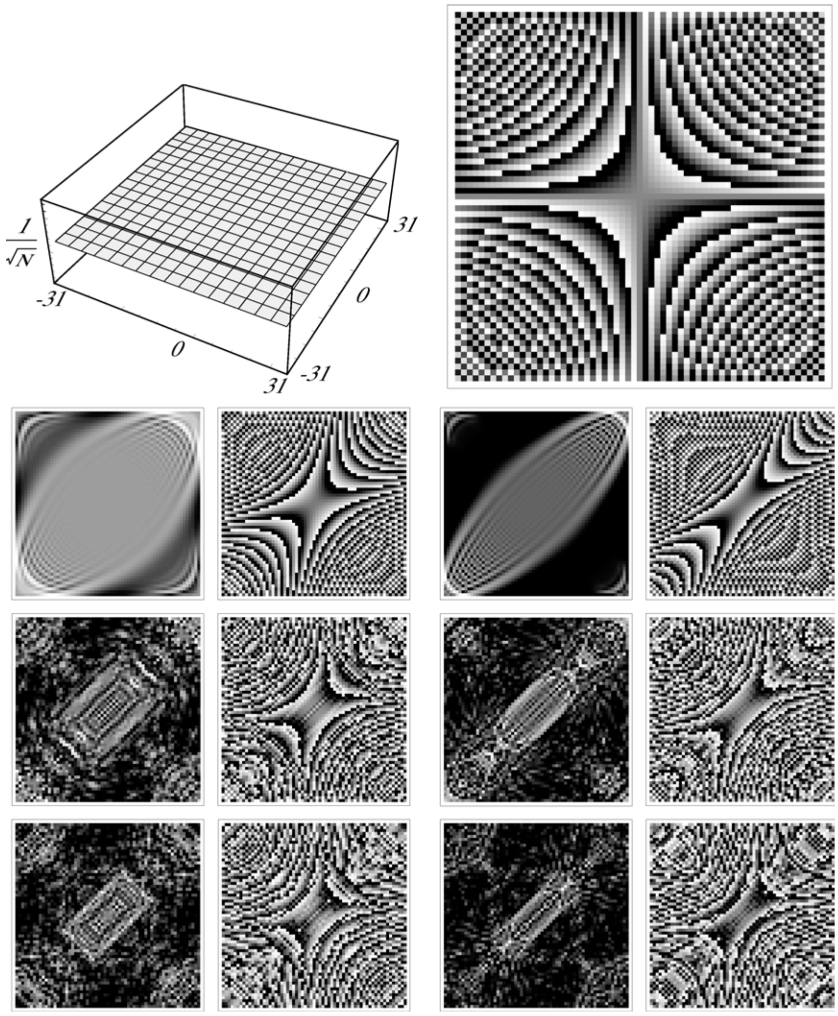

Figure 4. Absolute value and phase of the fractional discrete q-Fourier matrices $(\mathbf{F}_q^{\nu})_{m,m'}$ for $N = 63$. Upper row: the $\nu = 1$ DFT matrix common to all q. Lower-left columns: the q-FrDFT matrices for power $\nu = \frac{3}{4}$ and (from top to bottom) $q = 1$ (undeformed case), 0.9 and 0.8. Lower-right columns: the q-FrDFT matrices for power $\nu = \frac{1}{2}$ and the same values of q.

elements (figure 4), which served to compare the matrices with the fractional Fourier integral kernels (16).

The property of the integral kernels to have unit absolute value indicates that the action of the q-FrDFT matrices on signal vectors can be faithful to the integral transform only in the central range (small $|m|$) of the signal, as can be seen in figure 4, and this range diminishes with the power ν and the parameter q. In the same regions, the phases of the matrix elements do follow (roughly) the pattern of hyperbolas that characterize the fractional integral kernels. The q-FrDFTs of a rectangle signal were displayed (figure 5) for comparison with the same example in the extant literature.

We noted that the equivalent potential defined in (37) of the discrete system has the expected shape, resembling that of a harmonic oscillator (figure 3), and that it is independent of

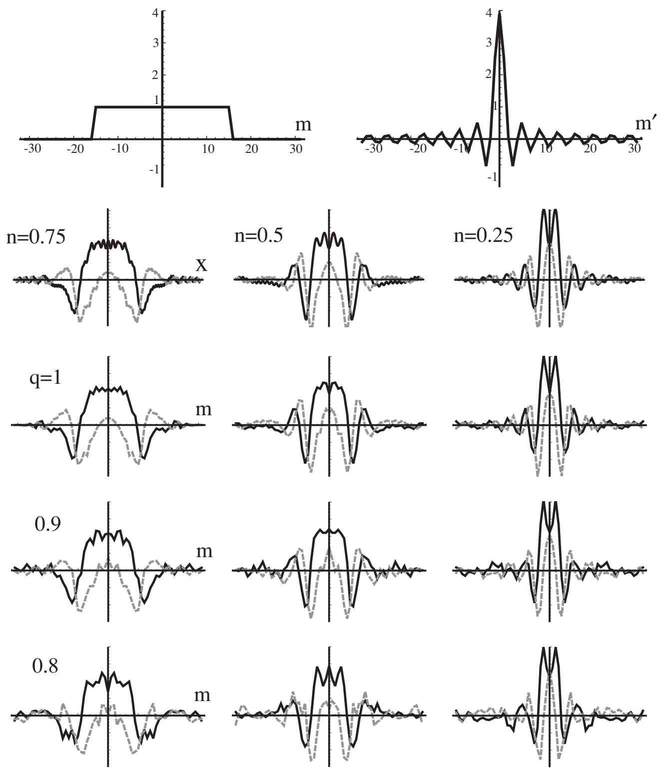

Figure 5. Fractional q-Fourier transforms of a rectangle signal with support on $(-15, 15)$ for $N = 63$. (Thick lines for real parts and dotted lines for imaginary parts.) Upper row: the rectangle signal ($\nu = 0$) and its DFT ($\nu = 1$) common to all q. Second row: continuous fractional FIT (15) of a rectangle function with the integral kernel (16) for powers (left to right) $\nu = \frac{3}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$. The scale factor between m and x is $\sqrt{(2\pi/N)}$. Last three rows: rectangle signals under the q-FrDFTs for $q = 1$ (undeformed case), 0.9 and 0.8, and the same powers ν.

the parameter q, so this deformation parameter cannot be contained in optical Fourier transform setups [23] that are distinct from the classical symmetric one-lens and other arrangements.

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